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# A CYLINDRICAL SHELL WITH AN ARBITRARILY ORIENTED CRACK†

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**Abstract**—In this paper the general problem of a shallow shell with constant curvatures is considered. It is assumed that the shell contains an arbitrarily oriented through crack and the material is specially orthotropic. The nonsymmetric problem is solved for arbitrary self-equilibrating crack surface tractions, which, added to an appropriate solution for an uncracked shell, would give the result for a cracked shell under most general loading conditions. The problem is reduced to a system of five singular integral equations in a set of unknown functions representing relative displacements and rotations on the crack surfaces. The stress state around the crack tip is asymptotically analyzed and it is shown that the results are identical to those obtained from the two-dimensional in-plane and anti-plane elasticity solutions. The numerical results are given for a cylindrical shell containing an arbitrarily oriented through crack. Some sample results showing the effect of the Poisson's ratio and the material orthotropy are also presented.

## 1. INTRODUCTION

Because of their potential applications to the strength and failure analysis of such structurally important elements as pressure vessels, pipes, and a great variety of aerospace and hydrospace components, in recent past the crack problems in shells have attracted considerable attention. Typical solutions obtained by using the classical shallow shell theory may be found, for example, in[1–4]. In a Mode I type of shell problem (that is, in a shell for which the geometry and the loading are symmetric with respect to the plane of the crack), particularly for membrane loading, the solution based on the classical theory seems to be adequate. However, in skewsymmetric or nonsymmetric problems, because of the Kirchoff assumption regarding the transverse shear and the twisting moment, in the classical solution it is not possible to separate Mode II and Mode III (i.e. respectively in-plane and anti-plane shear) stress states around the crack tips. In this case a singularity of the form $r^{-(1/2)}$ in Mode II stress state automatically implies $r^{-(3/2)}$ singularity in Mode III. For flat plates such drawbacks of the classical theory was pointed out in[5] where it was found that the asymptotic results obtained from plate bending and two-dimensional elasticity could be brought in agreement provided one uses a sixth order plate theory (e.g. that of Reissner's[6]).

In the crack problems for shells even though the membrane and bending results are coupled, the asymptotic behavior of the membrane and bending stresses around the crack tips should be identical to those given by respectively the plane stress and plate bending solutions. This was shown to be the case for the classical shell results (see, for example, the review article[7]). Recent studies using a Reissner-type shell theory[8, 9] show that similar agreement is also obtained between shell results and those given by the plane elasticity and a sixth order plate bending theory[10–13].

Because of the high likelihood of Mode I type fracture most of the previous studies of crack problems in shells were on the symmetrically loaded structures in which the crack is located in one of the principal planes of curvature. The advantage of this crack geometry is that one can always formulate the problems for one half of the shell only as a symmetric or an antisymmetric problem and reduce the number of unknowns. However, in such structural components as pipes and pipe elbows, if, in addition to internal pressure and bending the external loads include also torsion, then the most likely orientation of the crack initiation and propagation would be along a helix rather than a principal plane of curvature. In this case, the problem would have no symmetry and all five stress intensity factors associated with the five membrane, bending, and transverse shear resultants on the crack surface would be coupled. Consequently, the related mixed boundary value problem would reduce to a system of five pairs of dual integral equations or five singular integral equations.

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In this paper we consider the simplest and yet, from a practical viewpoint, perhaps the most important such problem, namely a cylindrical shell containing a through crack along an arbitrary direction with respect to the axis of the cylinder. In formulating the problem it is assumed that the regular solution of the shell without the crack for the given applied loads is obtained and the problem is reduced to a perturbation problem in which the self-equilibrating crack surface tractions are the only external loads.

## 2. THE BASIC EQUATIONS

As pointed out in the introduction, in solving the general nonsymmetric crack problems in shells it is necessary to use a transverse shear theory in order to obtain a singular behavior for the stress state around the crack tip which is consistent with the elasticity solutions of the in-plane and anti-plane crack problems. For shallow shells perhaps the simplest such theory would give the following equilibrium equations[9, 10]:

$$N_{ij,j} = 0, \tag{1}$$

$$V_{i,i} + (Z_{,i}N_{ij})_{,j} + q(x_1, x_2) = 0, \tag{2}$$

$$M_{ij,j} - V_i = 0, \ (i, j = 1, 2), \tag{3}$$

where $x_1$, $x_2$, $x_3$ are the rectangular coordinates, $x_1$, $x_2$ plane being tangent to the middle surface of the shell, $Z = Z(x_1, x_2)$ is the equation of the middle surface, $q$ is the transverse load, and $N_{ij}$, $M_{ij}$ and $V_i$, $(i,j = 1,2)$ are, respectively, the membrane, the moment and the transverse shear resultants. The displacement quantities corresponding to the stress resultants $N_{ij}$, $M_{ij}$ and $V_i$ are $(u_1, u_2)$, $(\beta_1, \beta_2)$ and $u_3$, respectively.

For an orthotropic shell† the stress–strain relations may be expressed as follows:

$$\epsilon_{11} = \frac{1}{E_1}(\sigma_{11} - \nu_1\sigma_{22}), \ \epsilon_{22} = \frac{1}{E_2}(\sigma_{22} - \nu_2\sigma_{11}), \ \epsilon_{12} = \sigma_{12}/2G_{12}, \ \nu_1/E_1 = \nu_2/E_2. \tag{4}$$

If we now assume that the material is specially orthotropic, the elastic constants satisfy the following factorization condition[12]:

$$G_{12} = \frac{(E_1E_2)^{1/2}}{2[1 + (\nu_1\nu_2)^{1/2}]}. \tag{5}$$

Thus, by defining the effective material constants as

$$E = (E_1E_2)^{1/2}, \ \nu = (\nu_1\nu_2)^{1/2}, \ G = E/2(1 + \nu), \ c = (E_1/E_2)^{1/4}, \tag{6}$$

and by observing that $\sigma_{ij} = N_{ij}/h$, where $h$ is the shell thickness, eqns (4) may be written as:

$$\epsilon_{11} = \frac{1}{hE}\left(\frac{N_{11}}{c^2} - \nu N_{22}\right), \ \epsilon_{22} = \frac{1}{hE}(c^2N_{22} - \nu N_{11}), \ \epsilon_{12} = \frac{1+\nu}{hE}N_{12}. \tag{7}$$

In shallow shells the in-plane components of the strains are given by

$$\epsilon_{ij} = \frac{1}{2}[u_{i,j} + u_{j,i} + Z_{,i}u_{3,j} + Z_{,j}u_{3,i}], \ (i,j = 1,2). \tag{8}$$

Similarly, the remaining stress resultant–displacement relations may be expressed as fol-

†The results given in[12] show that the effect of material orthotropy on the stress intensity factors can be quite significant. In practice the material may be orthotropic because it is either a composite laminate or a rolled sheet metal alloy. Orthotropic materials are also anisotropic with regard to their resistance to fracture and crack propagation. Hence in a cylindrical shell if the axes of orthotropy do not coincide with the axial and circumferential directions, the solution of the general inclined crack problem becomes all the more important. The solution is also necessary to analyze the weld defects and cracks initiated in the weak cleavage plane of the rolled sheet in spirally welded pipes.

lows:

$$M_{11} = D(c^2\beta_{1,1} + \nu\beta_{2,2}),\ M_{22} = D(\nu\beta_{1,1} + \beta_{2,2}/c^2),$$
$$M_{12} = \frac{D(1-\nu)}{2}(\beta_{1,2} + \beta_{2,1}); \tag{9}$$

$$V_1 = chB(u_{3,1} + \beta_1),\ V_2 = \frac{hB}{c}(u_{3,2} + \beta_2), \tag{10}$$

where ([8, 9])

$$D = \frac{Eh^3}{12(1-\nu^2)},\ B = \frac{5}{6}\frac{E}{2(1+\nu)}. \tag{11}$$

It may easily be seen that by substituting from (7)–(10) into (1)–(3) one obtains a system of five second order partial differential equations in the displacement quantities $u_1$, $u_2$, $u_3$, $\beta_1$ and $\beta_2$ which are equivalent to Navier's equations in elasticity. Thus, the unique solution of the system requires that five conditions be prescribed on the boundary. For example, on a boundary defined by $x_1$ = constant one function from each of the following five complementary pairs (or some suitable linear combinations of them) must be prescribed: $(N_{11}, u_1)$, $(N_{12}, u_2)$, $(M_{11}, \beta_1)$, $(M_{12}, \beta)$ and $(V_1, u_3)$.

## 3. THE CRACK PROBLEM

The particular crack problem under consideration is described in Fig. 1. Seemingly the problem appears to be quite complicated. However, by using the standard Fourier transforms it can be reduced to a system of integral equations in a relatively straightforward manner, Therefore, in this paper only the significant aspects of the analysis will be pointed out. For details of the analysis and for complete results we refer to reference[14].

By using the normalized quantities defined in Appendix A and a stress function $\phi$ the problem may be formulated as follows:

$$\nabla^4\phi - \frac{1}{\lambda^2}\left(\lambda_1^2\frac{\partial^2}{\partial y^2} - 2\lambda_{12}^2\frac{\partial^2}{\partial x\partial y} + \lambda_2^2\frac{\partial^2}{\partial x^2}\right)w = 0, \tag{12}$$

$$\nabla^4 w + \lambda^2(1-\kappa\nabla^2)\left(\lambda_1^2\frac{\partial^2}{\partial y^2} - 2\lambda_{12}^2\frac{\partial^2}{\partial x\partial y} + \lambda_2^2\frac{\partial^2}{\partial x^2}\right)\phi = \lambda^4(1-\kappa\nabla^2)\left(\frac{aq}{h}\right), \tag{13}$$

$$\kappa\nabla^2\psi - \psi - w = 0, \tag{14}$$

$$\frac{\kappa(1-\nu)}{2}\nabla^2\Omega - \Omega = 0. \tag{15}$$

The shell parameters $\lambda_1$, $\lambda_2$, $\lambda_{12}$, $\lambda$ and $\kappa$ are defined in Appendix A, $q(x, y)$ is the transverse

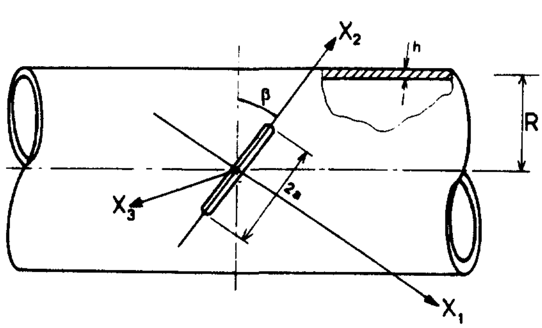


Fig. 1. Geometry of a cylindrical shell containing an inclined crack.

loading, and the curvatures are given by

$$\frac{1}{R_1} = -\frac{\partial^2 Z}{\partial x_1^2},\ \frac{1}{R_2} = -\frac{\partial^2 Z}{\partial x_2^2},\ \frac{1}{R_{12}} = -\frac{\partial^2 Z}{\partial x_1 \partial x_2}, \tag{16}$$

where $Z = Z(x_1, x_2)$ is the equation of the middle surface of the shell. The functions $\psi$ and $\Omega$ are related to the components of the rotation vector by

$$\beta_x = \frac{\partial \psi}{\partial x} + \kappa \frac{1-\nu}{2}\frac{\partial \Omega}{\partial y},\ \beta_y = \frac{\partial \psi}{\partial y} - \kappa \frac{1-\nu}{2}\frac{\partial \Omega}{\partial x}. \tag{17}$$

The normalized membrane, moment, and transverse shear resultants are given by

$$N_{xx} = \frac{\partial^2 \phi}{\partial y^2},\ N_{yy} = \frac{\partial^2 \phi}{\partial x^2},\ N_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y};$$

$$M_{xx} = \frac{a}{h\lambda^4}\left(\frac{\partial \beta_x}{\partial x} + \nu \frac{\partial \beta_y}{\partial y}\right),\ M_{yy} = \frac{a}{h\lambda^4}\left(\nu \frac{\partial \beta_x}{\partial x} + \frac{\partial \beta_y}{\partial y}\right), \tag{18}$$

$$M_{xy} = \frac{a}{h\lambda^4}\frac{1-\nu}{2}\left(\frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x}\right); \tag{19}$$

$$V_x = \frac{\partial w}{\partial x} + \beta_x,\ V_y = \frac{\partial w}{\partial y} + \beta_y. \tag{20}$$

The differential eqns (12)–(15) are solved by expressing the unknown functions in terms of Fourier integrals. As pointed out earlier, the system is equivalent to a tenth order partial differential equation. Also, the geometry of the shell indicates that the problem has no symmetry. Thus, the solution involves ten characteristic roots and ten "integration constants" for each half of the shell $x_1 > 0$ and $x_1 < 0$ (Fig. 1). Eight of the characteristic roots are found from

$$\begin{aligned} D(m) = {} & m^8 - (4\alpha^2 + \kappa\lambda_2^4)m^6 - 4\kappa\lambda_{12}^2\lambda_2^2\alpha i m^5 \\ & + [6\alpha^4 + \kappa(4\lambda_{12}^4 + 2\lambda_1^2\lambda_2^2 + \lambda_2^4)\alpha^2 + \lambda_2^4]m^4 \\ & + 4\lambda_{12}^2[\lambda_2^2 + \kappa(\lambda_1^2 + \lambda_2^2)\alpha^2]\alpha i m^3 \\ & - [4\alpha^4 + \kappa\alpha^2(4\lambda_{12}^4 + 2\lambda_1^2\lambda_2^2 + \lambda_1^4) + 4\lambda_{12}^4 + 2\lambda_1^2\lambda_2^2]\alpha^2 m^2 \\ & - 4\lambda_1^2\lambda_{12}^2(1 + \kappa\alpha^2)\alpha^3 i m + \alpha^4(\alpha^4 + \kappa\lambda_1^4\alpha^2 + \lambda_1^4) = 0, \end{aligned} \tag{21}$$

and two are given by

$$r_1 = -r_2 = -\left[\alpha^2 + \frac{2}{\kappa(1-\nu)}\right]^{1/2} \tag{22}$$

where $\alpha$ is the transform variable. In (21) by substituting $m = is$ it may be seen that

$$D(is) = \sum_0^8 a_k(\alpha)s^k = 0, \tag{23}$$

where the coefficients $a_k$ are real and hence the complex roots are in conjugate form. Since $\mathrm{Re}(m_j) = \mathrm{Im}(s_j)$, $(j = 1, \ldots, 8)$, by ordering the roots $m_j$ of (3.2) properly it may be shown that they have the following property:

$$\mathrm{Re}(m_{j+4}) = -\mathrm{Re}(m_j),\ \mathrm{Re}(m_j) < 0,\ (j = 1, \ldots, 4). \tag{24}$$

Since the problem under consideration is one of perturbation with the selfequilibrating crack surface tractions as the only external loads, because of the factors $\exp(m_j x)$ and $\exp(r_k x)$, $(j = 1, \ldots, 8;\ k = 1,2)$, associated with each integration constant, for each half of the shell only five constants must be retained in the analysis which correspond to $m_1, \ldots, m_4$ and $r_1$ for $x_1 > 0$ and $m_5, \ldots, m_8$ and $r_2$ for $x < 0$. The problem may then be reduced to the determination of ten unknowns, $R_1(\alpha), \ldots, R_4(\alpha)$ and $A_1(\alpha)$ for $x_1 > 0$ and $R_5(\alpha), \ldots, R_8(\alpha)$ and $A_2(\alpha)$ for $x_1 < 0$.

In the crack problem these ten unknowns are obtained by using the following continuity and boundary conditions:

$$N_{xx}(+0, y) = N_{xx}(-0, y), \quad -\infty < y < \infty, \tag{25}$$

$$M_{xx}(+0, y) = M_{xx}(-0, y), \quad -\infty < y < \infty, \tag{26}$$

$$N_{xy}(+0, y) = N_{xy}(-0, y), \quad -\infty < y < \infty, \tag{27}$$

$$M_{xy}(+0, y) = M_{xy}(-0, y), \quad -\infty < y < \infty, \tag{28}$$

$$V_x(+0, y) = V_x(-0, y), \quad -\infty < y < \infty; \tag{29}$$

$$\left.\begin{aligned} N_{xx}(+0, y) &= F_1(y), \quad |y| < \sqrt{(c)}, \\ u(+0, y) - u(-0, y) &= 0, \quad |y| > \sqrt{(c)}, \end{aligned}\right\} \tag{30a,b}$$

$$\left.\begin{aligned} M_{xx}(+0, y) &= F_2(y), \quad |y| < \sqrt{(c)}, \\ \beta_x(+0, y) - \beta_x(-0, y) &= 0, \quad |y| > \sqrt{(c)}, \end{aligned}\right\} \tag{31a,b}$$

$$\left.\begin{aligned} N_{xy}(+0, y) &= F_3(y), \quad |y| < \sqrt{(c)}, \\ v(+0, y) - v(-0, y) &= 0, \quad |y| > \sqrt{(c)}, \end{aligned}\right\} \tag{32a,b}$$

$$\left.\begin{aligned} M_{xy}(+0, y) &= F_4(y), \quad |y| < \sqrt{(c)}, \\ \beta_y(+0, y) - \beta_y(-0, y) &= 0, \quad |y| > \sqrt{(c)}, \end{aligned}\right\} \tag{33a,b}$$

$$\left.\begin{aligned} V_x(+0, y) &= F_5(y), \quad |y| < \sqrt{(c)}, \\ w(+0, y) - w(-0, y) &= 0, \quad |y| > \sqrt{(c)}, \end{aligned}\right\} \tag{34a,b}$$

where $F_1, \ldots, F_5$ are the known crack surface loads.

All the field quantities which appear in (25)–(34) may be expressed in terms of the unknown functions $R_1, \ldots, R_8$, $A_1$ and $A_2$. It is, therefore, clear that the homogeneous conditions (25)–(29) may be used to eliminate five of the unknowns. The remaining five are then obtained from the mixed boundary conditions (30)–(34). The mixed boundary conditions may be reduced to either a system of dual integral equations in the transform domain or a system of integral equations in the physical domain. As in many complicated mixed boundary value problems, in this case, too, the latter approach is by far the simpler of the two. From (30)–(34) it is seen that the mixed boundary conditions are given in terms of complementary stress and displacement quantities and the displacement quantities would be the natural new unknown functions in the system of integral equations to be derived. However, in order to avoid kernels with strong singularities in the resulting integral equations, it is necessary that the new unknown functions be selected as the derivatives of the displacement quantities rather than displacements and rotations themselves. Of the displacement quantities which appear in the mixed boundary conditions, $\beta_x$, $\beta_y$ and $w$ can be expressed in terms of $A_j$ and $R_i$ directly. However, to obtain $u$ and $v$ eqns (7), (8) and (18) must be used. Because of this, the selection of $\partial(u^+ - u^-)/\partial y$ and $\partial(v^+ - v^-)/\partial y$ (which would otherwise have been the natural choice) as the new unknown functions becomes extremely cumbersome and more suitable combinations are needed. In the present problem the derivation of the integral equations and the asymptotic analysis become relatively simple if the complementary displacement quantities (which are the new unknowns) are selected as follows:

$$G_1(y) = \lim_{x\to+0}\left[\frac{\partial u}{\partial y} - \left(\frac{\lambda_{12}}{\lambda}\right)^2 \int y \frac{\partial^2 w}{\partial y^2}\,dy\right] - \lim_{x\to-0}\left[\frac{\partial u}{\partial y} - \left(\frac{\lambda_{12}}{\lambda}\right)^2 \int y \frac{\partial^2 w}{\partial y^2}\,dy\right], \tag{35}$$

$$G_2(y) = \lim_{x\to+0} \frac{\partial \beta_x}{\partial y} - \lim_{x\to-0} \frac{\partial \beta_x}{\partial y}, \tag{36}$$

$$G_3(y) = \lim_{x\to+0}\left[\frac{\partial v}{\partial y} - \left(\frac{\lambda_2}{\lambda}\right)^2 y \frac{\partial w}{\partial y}\right] - \lim_{x\to-0}\left[\frac{\partial v}{\partial y} - \left(\frac{\lambda_2}{\lambda}\right)^2 y \frac{\partial w}{\partial y}\right], \tag{37}$$

$$G_4(y) = \lim_{x\to+0} \frac{\partial \beta_y}{\partial y} - \lim_{x\to-0} \frac{\partial \beta_y}{\partial y}, \quad (38)$$

$$G_5(y) = \lim_{x\to+0} \frac{\partial w}{\partial y} - \lim_{x\to-0} \frac{\partial w}{\partial y}. \quad (39)$$

By using (25)–(29) and (35)–(39) the unknowns $R_j(\alpha)$ and $A_j(\alpha)$ may be obtained as,

$$R_j(\alpha) = \sum_{k=1}^{5} iB_{jk}(\alpha)g_k(\alpha), \ (j = 1,\ldots,8), \quad (40)$$

$$A_j(\alpha) = \sum_{k=1}^{5} C_{jk}(\alpha)g_k(\alpha), \ (j = 1, 2), \quad (41)$$

where $B_{jk}$ and $C_{jk}$ are known functions[14] and

$$g_k(\alpha) = \int_{-\surd(c)}^{\surd(c)} G_k(t)\, e^{i\alpha t}\, dt, \ (k = 1,\ldots,5). \quad (42)$$

From the conditions (30b)–(34b) it may be observed that $G_k(y) = 0$, $(k = 1,\ldots,5)$ for $|y| > \surd(c)$. However, further conditions must be imposed on $G_k$ in order to insure the continuity of displacements and rotations in the shell along $x = 0$, $|y| > \surd(c)$. Clearly, $G_k(y)$ must be such that

$$\int_{-\surd(c)}^{\surd(c)} \frac{\partial}{\partial y}[\omega_k(+0, y) - \omega_k(-0, y)]\, dy = 0, \ (k = 1,\ldots,5), \quad (43)$$

where $\omega_1,\ldots,\omega_5$ represent the displacement quantities $u$, $v$, $\beta_x$, $\beta_y$ and $w$. From (35)–(39) it may be shown that the single-valuedness conditions are satisfied if

$$\int_{-\surd(c)}^{\surd(c)} \left[G_1(t) + \left(\frac{\lambda_{12}}{\lambda}\right)^2 tG_5(t)\right] dt - \left(\frac{\lambda_{12}}{\lambda}\right)^2 \int_{-\surd(c)}^{\surd(c)} dt \int_{-\surd(c)}^{t} G_5(y)\, dy = 0, \quad (44)$$

$$\int_{-\surd(c)}^{\surd(c)} G_2(t)\, dt = 0, \quad (45)$$

$$\int_{-\surd(c)}^{\surd(c)} \left[G_3(t) + \left(\frac{\lambda_2}{\lambda}\right)^2 tG_5(t)\right] dt = 0, \quad (46)$$

$$\int_{-\surd(c)}^{\surd(c)} G_4(t)\, dt = 0, \quad (47)$$

$$\int_{-\surd(c)}^{\surd(c)} G_5(t)\, dt = 0. \quad (48)$$

With the conditions (44)–(48) and the requirement that $G_k(y)$ be zero for $|y| > \surd(c)$, the second part (30b)–(34b) of the mixed boundary conditions is satisfied. The first part (30a)–(34a) relating to the crack surface loading would then give the integral equations to determine $G_1,\ldots,G_5$. After some lengthy asymptotic analysis it can be shown that the kernels of the resulting integral equations have simple Cauchy type singularities which may readily be separated and the integral equations may be expressed as follows:

$$\int_{-\surd(c)}^{\surd(c)} \left[\frac{G_1(t)}{t-y} + \sum_{1}^{5} k_{1j}(y,t)G_j(t)\right] dt = 4\pi F_1(y), |y| < \surd(c), \quad (49)$$

$$\int_{-\surd(c)}^{\surd(c)} \left[\frac{1-\nu^2}{\lambda^4}\frac{G_2(t)}{t-y} + \sum_{1}^{5} k_{2j}(y,t)G_j(t)\right] dt = 4\pi\frac{h}{a}F_2(y), \ |y| < \surd(c), \quad (50)$$

$$\int_{-\surd(c)}^{\surd(c)} \left[\frac{G_3(t)}{t-y} + \sum_{1}^{5} k_{3j}(y,t)G_j(t)\right] dt = 4\pi F_3(y), \ |y| < \surd(c), \quad (51)$$

$$\int_{-\sqrt{(c)}}^{\sqrt{(c)}}\left[\frac{1-\nu^2}{\lambda^4}\frac{G_4(t)}{t-y}+\sum_1^5 k_{4j}(y,t)G_j(t)\right]dt = 4\pi\frac{h}{a}F_4(y),\ |y|<\sqrt{(c)}, \tag{52}$$

$$\int_{-\sqrt{(c)}}^{\sqrt{(c)}}\left[\frac{G_5(t)}{t-y}+\sum_1^5 k_{5j}(y,t)G_j(t)\right]dt = 4\pi F_5(y),\ |y|<\sqrt{(c)}. \tag{53}$$

The expressions of the Fredholm kernels, $k_{ij}(y, t)$, $(i,j = 1,\ldots,5)$, as well as details of the analysis may be found in[14]. The system of singular integral eqns (49)–(53) must be solved under the additional conditions (44)–(48). They may be solved in a straightforward manner by using the Gaussian integration technique (see, for example, [15]). The major work in this problem is the evaluation of the Fredholm kernels $k_{ij}(y, t)$, $(i, j = 1,\ldots,5)$ which are given in terms of Fourier integrals. To improve the accuracy the asymptotic parts of all integrands are separated and the related integrals are evaluated in closed form. The details of this analysis may also be found in[14].

## 4. ASYMPTOTIC STRESS FIELD AROUND THE CRACK TIPS

For the numerical solution of the system of singular integral eqns (49)–(53) the interval $(-\sqrt{(c)}, \sqrt{(c)})$ is normalized by defining

$$\tau = t/\sqrt{(c)},\ \eta = y/\sqrt{(c)},\ \xi = x/\sqrt{(c)},$$
$$H_j(\tau) = G_j(\tau\sqrt{(c)}),\ (j = 1,\ldots,5),\ -1<\tau<1. \tag{54}$$

We now observe that the index of the system of singular integral equations is +1 and its solution is of the following form:

$$H_j(\tau) = h_j(\tau)/(1-\tau^2)^{1/2},\ (-1<\tau<1),\ (j = 1,\ldots,5), \tag{55}$$

where $h_1,\ldots,h_5$ are unknown bounded functions. Going back to the original formulation of the problem and by using (40)–(43), (54) and (55), after again some lengthy asymptotic analysis, in a small neighborhood of the crack tip ($\xi = 1, \eta = 0$) the stress resultants may be expressed as follows:

$$N_{xx} \cong -\frac{h_3(1)}{4\sqrt{(2r)}}\left[-\frac{1}{4}\sin\frac{\theta}{2}+\frac{1}{4}\sin\frac{5\theta}{2}\right]-\frac{h_1(1)}{4\sqrt{(2r)}}\left[\frac{5}{4}\cos\frac{\theta}{2}-\frac{1}{4}\cos\frac{5\theta}{2}\right], \tag{56}$$

$$N_{yy} \cong -\frac{h_3(1)}{4\sqrt{(2r)}}\left[-\frac{7}{4}\sin\frac{\theta}{2}-\frac{1}{4}\sin\frac{5\theta}{2}\right]-\frac{h_1(1)}{4\sqrt{(2r)}}\left[\frac{3}{4}\cos\frac{\theta}{2}+\frac{1}{4}\cos\frac{5\theta}{2}\right], \tag{57}$$

$$N_{xy} \cong -\frac{h_3(1)}{4\sqrt{(2r)}}\left[\frac{3}{4}\cos\frac{\theta}{2}+\frac{1}{4}\cos\frac{5\theta}{2}\right]-\frac{h_1(1)}{\sqrt{(2r)}}\left[-\frac{1}{4}\sin\frac{\theta}{2}+\frac{1}{4}\sin\frac{5\theta}{2}\right], \tag{58}$$

$$M_{xx} \cong -\frac{h_4(1)}{4\sqrt{(2r)}}\frac{h}{12a}\left[-\frac{1}{4}\sin\frac{\theta}{2}+\frac{1}{4}\sin\frac{5\theta}{2}\right]-\frac{h_2(1)}{4\sqrt{(2r)}}\frac{h}{12a}\left[\frac{5}{4}\cos\frac{\theta}{2}-\frac{1}{4}\cos\frac{5\theta}{2}\right], \tag{59}$$

$$M_{yy} \cong -\frac{h_4(1)}{4\sqrt{(2r)}}\frac{h}{12a}\left[-\frac{7}{4}\sin\frac{\theta}{2}-\frac{1}{4}\sin\frac{5\theta}{2}\right]-\frac{h_2(1)}{4\sqrt{(2r)}}\frac{h}{12a}\left[\frac{3}{4}\cos\frac{\theta}{2}+\frac{1}{4}\cos\frac{5\theta}{2}\right], \tag{60}$$

$$M_{xy} \cong -\frac{h_4(1)}{4\sqrt{(2r)}}\frac{h}{12a}\left[\frac{3}{4}\cos\frac{\theta}{2}+\frac{1}{4}\cos\frac{5\theta}{2}\right]-\frac{h_2(1)}{4\sqrt{(2r)}}\frac{h}{12a}\left[-\frac{1}{4}\sin\frac{\theta}{2}+\frac{1}{4}\sin\frac{5\theta}{2}\right], \tag{61}$$

$$V_x \cong -\frac{h_5(1)}{2\sqrt{(2r)}}\cos\frac{\theta}{2}, \tag{62}$$

$$V_y \cong \frac{h_5(1)}{2\sqrt{(2r)}}\sin\frac{\theta}{2}, \tag{63}$$

where $r$, $\theta$ are the polar coordinates at the crack tip defined by

$$r\sin\theta = \xi,\ r\cos\theta = \eta - 1. \tag{64}$$

By observing that the membrane and bending components of the stresses are given by (see Appendix A)

$$\sigma_{ij}^{m} = N_{ij}, \quad \sigma_{ij}^{b} = \frac{12az}{h} M_{ij} \quad (i,j = x,y) \tag{65}$$

from eqns (56)–(61) for the leading terms of the combined in-plane stresses $\sigma_{ij} = \sigma_{ij}^{m} + \sigma_{ij}^{b}$, $(i,j = x,y)$ we obtain

$$\sigma_{xx} \cong -\frac{h_3(1)+zh_4(1)}{4\sqrt{(2r)}}\left[-\frac{1}{4}\sin\frac{\theta}{2}+\frac{1}{4}\sin\frac{5\theta}{2}\right]-\frac{h_1(1)+zh_2(1)}{4\sqrt{(2r)}}\left[\frac{5}{4}\cos\frac{\theta}{2}-\frac{1}{4}\cos\frac{5\theta}{2}\right], \tag{66}$$

$$\sigma_{yy} \cong -\frac{h_3(1)+zh_4(1)}{4\sqrt{(2r)}}\left[-\frac{7}{4}\sin\frac{\theta}{2}-\frac{1}{4}\sin\frac{5\theta}{2}\right]-\frac{h_1(1)+zh_2(1)}{4\sqrt{(2r)}}\left[\frac{3}{4}\cos\frac{\theta}{2}+\frac{1}{4}\cos\frac{5\theta}{2}\right], \tag{67}$$

$$\sigma_{xy} \cong -\frac{h_3(1)+zh_4(1)}{4\sqrt{(2r)}}\left[\frac{3}{4}\cos\frac{\theta}{2}+\frac{1}{4}\cos\frac{5\theta}{2}\right]-\frac{h_1(1)+zh_2(1)}{4\sqrt{(2r)}}\left[-\frac{1}{4}\sin\frac{\theta}{2}+\frac{1}{4}\sin\frac{5\theta}{2}\right]. \tag{68}$$

Similarly, for the transverse shear stresses from

$$\sigma_{iz} = \frac{3}{2} V_i\left[1-\left(\frac{az}{h/2}\right)^2\right], \quad (i = x,y), \tag{69}$$

we obtain

$$\sigma_{xz} \cong -\frac{3}{2}\frac{h_5(1)}{2\sqrt{(2r)}}\cos\frac{\theta}{2}\left[1-\left(\frac{az}{h/2}\right)^2\right], \tag{70}$$

$$\sigma_{yz} \cong -\frac{3}{2}\frac{h_5(1)}{2\sqrt{(2r)}}\sin\frac{\theta}{2}\left[1-\left(\frac{az}{h/2}\right)^2\right], \tag{71}$$

Note that for the isotropic materials $c = 1$ and the asymptotic stress fields (66)–(68) and (70)–(71) found from the shell solution are identical to those given by respectively the in-plane and the anti-plane elasticity solutions of a two-dimensional crack problem. If we now define the Modes I, II and III stress intensity factors (for a crack along $x_1 = 0$, $-a < x_2 < a$) by

$$k_j(x_3) = \lim_{x_2 \to a} \sqrt{(2(x_2 - a))}\,\sigma_{1j}(0, x_2, x_3), \quad (j = 1, 2, 3), \tag{72}$$

from (66)–(68) and (70) and Appendix A we obtain

$$k_1(x_3) = -\frac{cE}{4}\sqrt{(a)}\left[h_1(1)+\frac{x_3}{a}h_2(1)\right], \tag{73}$$

$$k_2(x_3) = -\frac{E\sqrt{(a)}}{4}\left[h_3(1)+\frac{x_3}{a}h_4(1)\right], \tag{74}$$

$$k_3(x_3) = -\frac{3}{4}B\sqrt{(a)}\sqrt{(c)}h_5(1)\left[1-\left(\frac{x_3}{h/2}\right)^2\right]. \tag{75}$$

## 5. THE RESULTS AND DISCUSSION

The main interest in this study is in evaluating the stress intensity factors in shells for various crack geometries and loading conditions. For each crack geometry the problem is solved by assuming only one of the five possible crack surface loadings to be nonzero at a time. For a general loading the results may then be obtained by superposition. From (73) and (74) it is seen that the in-plane stress intensity factors $k_1$ and $k_2$ have a "membrane" and a "bending" component, and $h_1$ and $h_3$ are related to the membrane and $h_2$ and $h_4$ are related to the bending stresses. For simplicity, the related stress intensity factors are defined separately. The calculated results are normalized with respect to a standard stress intensity factor $\sigma_j\sqrt{(a)}$ where $\sigma_j$ stands for any one of the following five nominal ("membrane", "bending", in-plane "shear", "twisting", and "transverse shear") stresses:

$$\sigma_m = N_{11}/h,\ \sigma_b = 6M_{11}/h^2,\ \sigma_s = N_{12}/h,\ \sigma_t = 6M_{12}/h^2,\ \sigma_v = (3/2)V_1/h, \tag{76}$$

where crack lies in $x_2x_3$ plane and $N_{11}$, $M_{11}$, $N_{12}$, $M_{12}$ and $V_1$ are (a measure or amplitude of) the crack surface tractions.

The normalized stress intensity factors are then defined and calculated in terms of $h_i(1)$, $(i = 1,\ldots,5)$ as follows:

$$k_{mj} = \frac{k_1(0)}{\sigma_j\sqrt{(a)}} = -\frac{cE}{4\sigma_j}h_1(1), \tag{77}$$

$$k_{bj} = \frac{k_1(h/2) - k_1(0)}{\sigma_j\sqrt{(a)}} = -\frac{cE}{4\sigma_j}\frac{h}{2a}h_2(1), \tag{78}$$

$$k_{sj} = \frac{k_2(0)}{\sigma_j\sqrt{(a)}} = -\frac{E}{4\sigma_j}h_3(1), \tag{79}$$

$$k_{tj} = \frac{k_2(h/2) - k_2(0)}{\sigma_j\sqrt{(a)}} = -\frac{E}{4\sigma_j}\frac{h}{2a}h_4(1), \tag{80}$$

$$k_{vj} = \frac{k_3(0)}{\sigma_i\sqrt{(a)}} = -\frac{3}{4}\frac{B}{\sigma_j}\sqrt{(c)}\,h_5(1),\ (j = m, b, s, t, v), \tag{81}$$

where for each individual loading $\sigma_j$ is given by (76). In the case of uniform crack surface loads $N_{11}$, $M_{11}$, $N_{12}$, $M_{12}$ and $V_1$, referring to (30a)–(34a), Appendix A, and (76) the input functions of the system of integral eqns (49)–(53) are given by

$$F_1(y) = \frac{\sigma_m}{cE},\ F_2(y) = \frac{\sigma_b}{6cE},\ F_3(y) = \frac{\sigma_s}{E},\ F_4(y) = \frac{\sigma_t}{6E},\ F_5(y) = \frac{2}{3}\frac{\sigma_v}{B\sqrt{(c)}}. \tag{82}$$

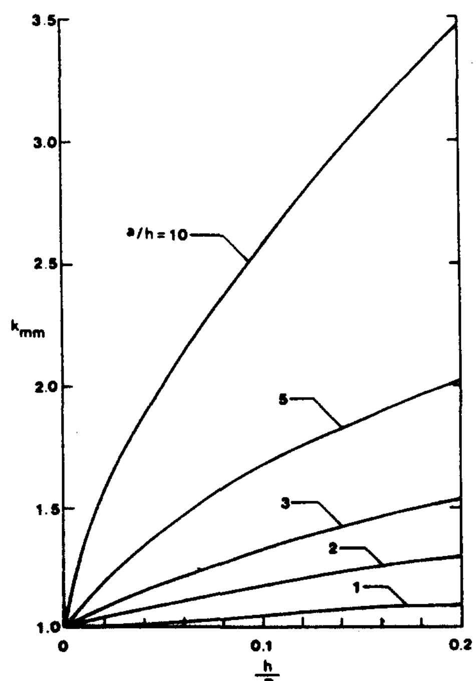


Fig. 2. Stress intensity factor ratio $k_{mm}$ in an isotropic cylindrical shell containing an inclined crack under uniform membrane loading $N_{11}$; $\beta = 45°$, $\nu = 0.3$.

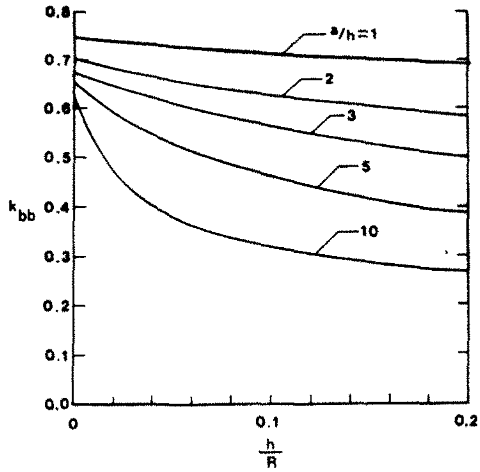


Fig. 3. Stress intensity factor ratio $k_{bb}$ in an isotropic cylindrical shell containing an inclined crack under uniform bending moment $M_{11}$; $\beta = 45°$, $\nu = 0.3$.

Even though the formulation given in this paper is valid for any shell with constant curvatures $1/R_1$, $1/R_2$ and $1/R_{12}$, the results are obtained for the practical problem of a cylindrical shell containing an arbitrarily oriented crack only (Fig. 1). The crack is assumed to be in a plane defined by the angle $\beta$ shown in Fig. 1. For the shallow cylindrical shell the curvatures referred to $x_1$, $x_2$ axes shown in the figure and defined by (16) may be expressed as

$$\frac{1}{R_1} = \frac{\sin^2\beta}{R}, \quad \frac{1}{R_2} = \frac{\cos^2\beta}{R}, \quad \frac{1}{R_{12}} = -\frac{\sin\beta\cos\beta}{R}. \tag{83}$$

Some numerical results obtained for an isotropic cylinder are shown in Figs. 2–11. Figures 2–6 show the primary stress intensity factor ratios $k_{mm}$, $k_{bb}$, $k_{ss}$, $k_{tt}$ and $k_{vv}$ for a cylinder having

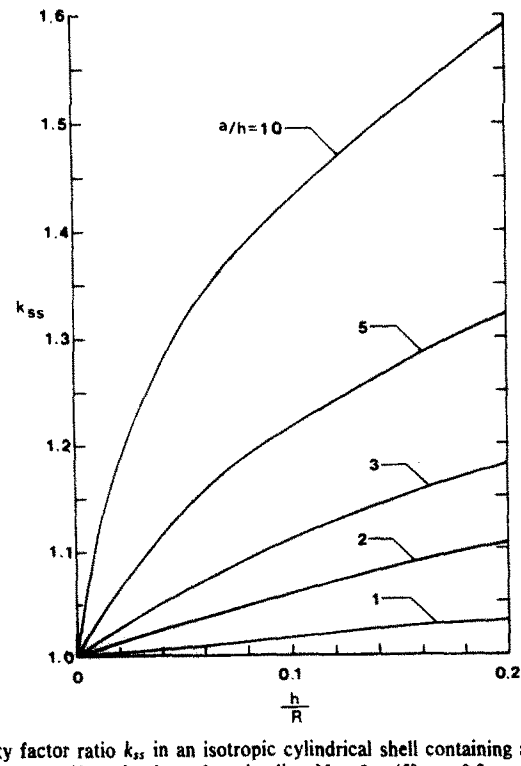


Fig. 4. Stress intensity factor ratio $k_{ss}$ in an isotropic cylindrical shell containing an inclined crack under uniform in-plane shear loading $N_{12}$; $\beta = 45°$, $\nu = 0.3$.

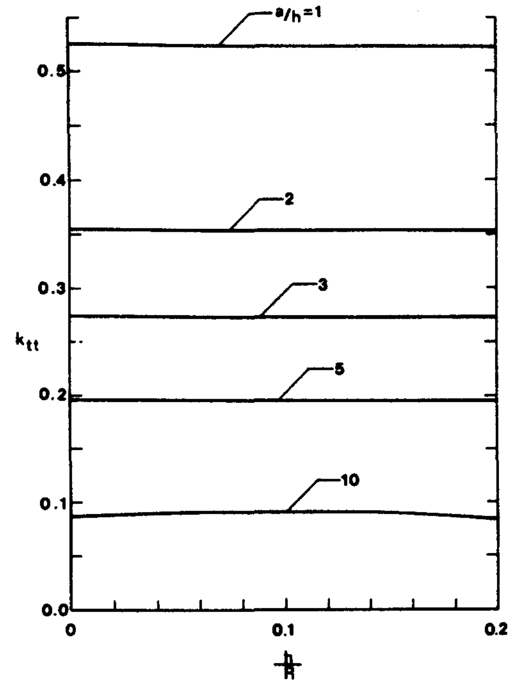


Fig. 5. Stress intensity factor ratio $k_{tt}$ in an isotropic cylindrical shell containing an inclined crack under uniform twisting moment $M_{12}$; $\beta = 45°$, $\nu = 0.3$.

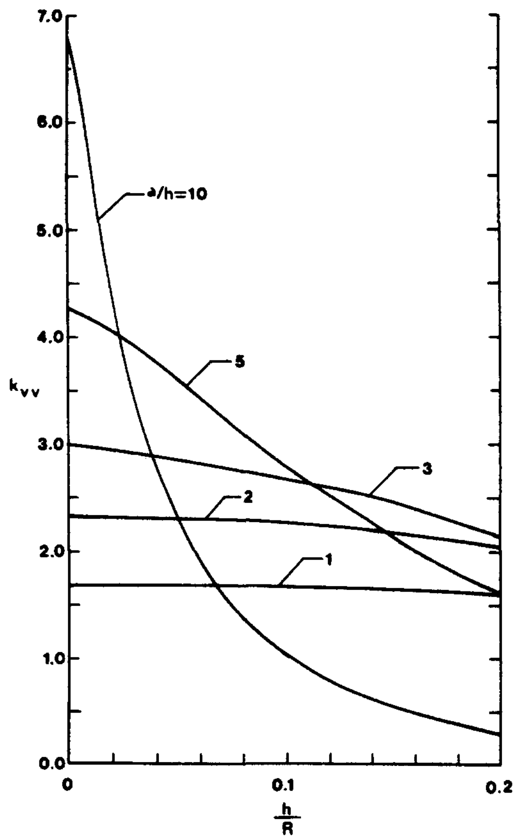


Fig. 6. Stress intensity factor ratio $k_{vv}$ in an isotropic cylindrical shell containing an inclined crack under uniform transverse shear loading $V_1$; $\beta = 45°$, $\nu = 0.3$.

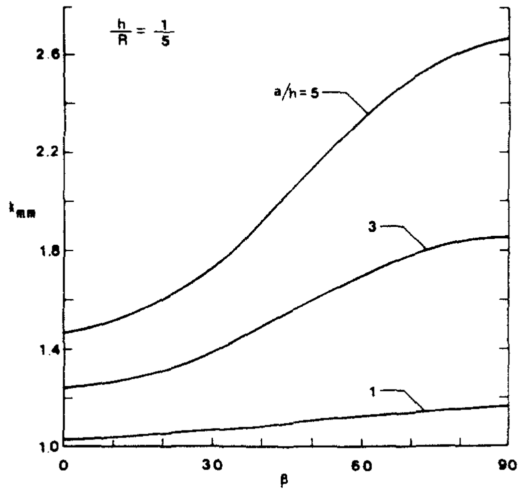


Fig. 7. Stress intensity factor ratio $k_{mm}$ in an isotropic cylindrical shell containing an inclined crack under uniform membrane loading $N_{11}$; $\nu = 0.3$, $h/R = 1/5$.

a crack inclined 45° with respect to the axis (see (77)–(81) and (73)–(75)). The unusual results here are those found for $k_{tt}$ and $k_{vv}$. Under a twisting moment $M_{12}$ uniformly distributed along the crack, the Mode II stress intensity factor ratio $k_{tt}$ appears to be nearly independent of the shell curvature $1/R$ but highly dependent on $a/h$. Figure 6 shows that the monotonic variation of the stress intensity factor ratios with $a/h$ and $h/R$ observed in Figs. 2–5 and in previous shell solutions is not valid for $k_{vv}$. This seems to be the case for all values of $\beta$ varying from zero to ninety degrees.

The effect of $\beta$ on the primary stress intensity ratios $k_{mm}$, $k_{bb}$, $k_{ss}$, $k_{tt}$ and $k_{vv}$ is shown in Figs. 7–11. Extensive results giving all stress intensity ratios $k_{ij}$ $(i, j = m, b, s, t, v)$ for $\beta = 0$, 15°, 30°, 45°, 60°, 75°, 90° and for varying $h/R$ and $a/h$ may be found in[14]. Table 1 shows some sample results regarding the secondary stress intensity ratios in a cylinder with a 45° crack under torsion (i.e. $N_{11}$ = constant and all other crack surface tractions zero).

The stress intensity factors given in Figs. 2–11 and in Table 1 are obtained for the Poisson's ratio $\nu = 0.3$. Some sample results showing the effect of $\nu$ on the stress intensity factors are given in Table 2. It is seen that this effect is not really significant.

It should be noted that the Poisson's ratio $\nu$ in isotropic shells and $\nu = \sqrt{(\nu_1 \nu_2)}$ and the

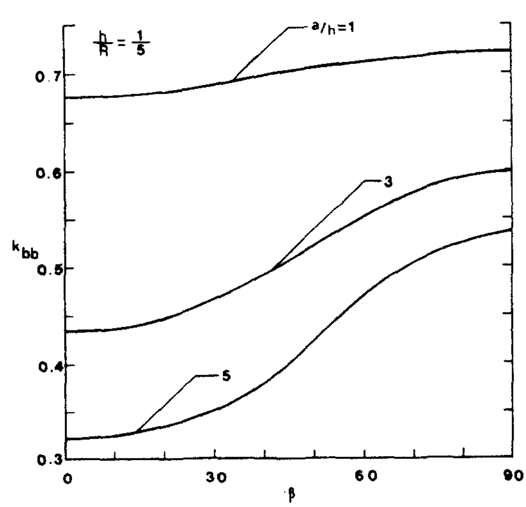


Fig. 8. Stress intensity factor ratio $k_{bb}$ in an isotropic cylindrical shell containing an inclined crack under uniform bending moment $M_{11}$; $\nu = 0.3$, $h/R = 1/5$.

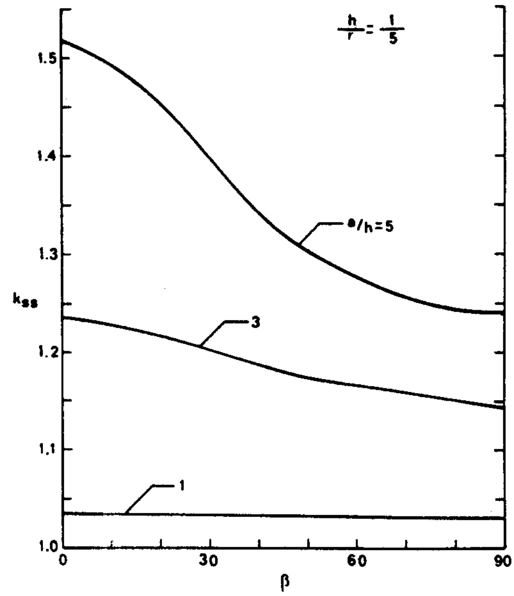


Fig. 9. Stress intensity factor ratio $k_{ss}$ in an isotropic cylindrical shell containing an inclined crack under uniform in-plane shear loading $N_{12}$; $\nu = 0.3$, $h/R = 1/5$.

stiffness ratio $c = (E_1/E_2)^{1/4}$ in specially orthotropic shells appear in the expressions of the kernels of the integral equations. Thus, to investigate the effect of the material orthotropy on the stress intensity factors both $\nu$ and $c$ must be varied. However, as seen from Table 2 the influence of $\nu$ is rather insignificant. Therefore, to study the effect of the material orthotropy it may be sufficient to vary $c$ only. For a strongly orthotropic material (in this case a graphite-epoxy composite) this effect is shown in Table 3. The axes of material orthotropy are along 45° directions with respect to the cylinder axis and the crack is located along one or the other axis of orthotropy. The Poisson's ratio is $\nu = \sqrt{(\nu_1 \nu_2)} = 0.037$ for the orthotropic shells and $\nu = 0.3$ for the isotropic results which are included for the purpose of comparison. The table shows that the effect of material orthotropy on the stress intensity factors could be very significant.

The quantity which is of some interest in certain fracture studies is the rate of internally

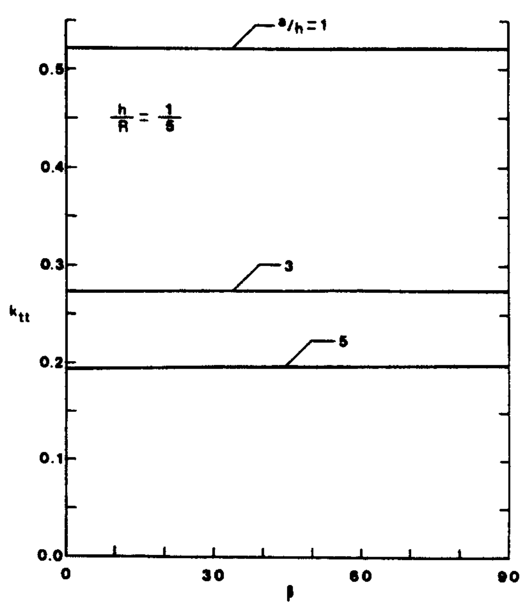


Fig. 10. Stress intensity factor ratio $k_{tt}$ in an isotropic cylindrical shell containing an inclined crack under uniform twisting moment $M_{12}$; $\nu = 0.3$, $h/R = 1/5$.

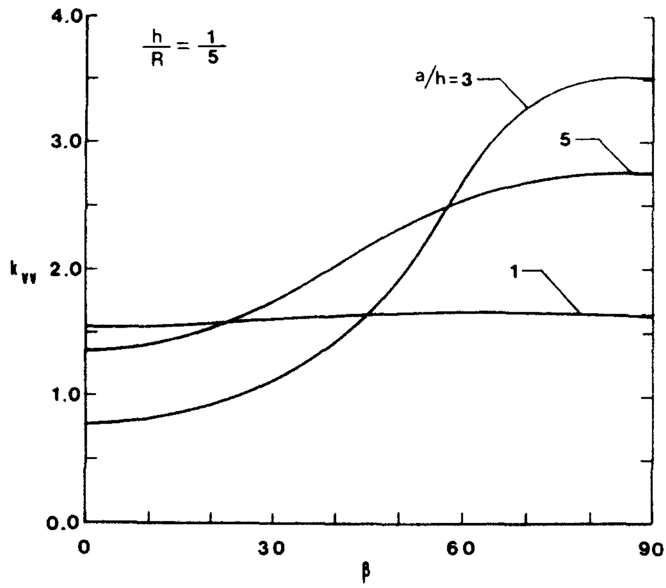


Fig. 11. Stress intensity factor ratio $k_{vv}$ in an isotropic cylindrical shell containing an inclined crack under uniform transverse shear loading $V_1$; $\nu = 0.3$, $h/R = 1/5$.

released or externally added energy per unit fracture area created as a result of crack propagation. If $U$ is the work of the external loads, $V$ the total strain energy, and $A$ the fracture surface, then in a quasistatic problem the rate of energy available for fracture would be $d(U - V)/dA$. For elastic problems this energy rate is known to be the same for "fixed grip" and "fixed load" conditions. It can therefore be calculated as the crack closure energy under fixed grip conditions. Under these conditions, $dU = 0$ and for a crack going from $x_2 = a$ to $x_2 = a + da$,

Table 1. Stress intensity factor ratios in an isotropic cylindrical shell containing an inclined crack under uniform membrane loading $N_{11}$; $\nu = 0.3$, $\beta = 45°$

| | $h/R$ \ $a/h$ | 1 | 2 | 3 | 5 | 10 |
|---|---|---|---|---|---|---|
| $k_{mm}$ | 1/5 | 1.097 | 1.302 | 1.544 | 2.030 | 3.486 |
| | 1/10 | 1.049 | 1.167 | 1.321 | 1.665 | 2.516 |
| | 1/15 | 1.033 | 1.116 | 1.230 | 1.501 | 2.199 |
| | 1/25 | 1.020 | 1.072 | 1.148 | 1.341 | 1.886 |
| | 1/50 | 1.010 | 1.037 | 1.079 | 1.194 | 1.563 |
| | 1/100 | 1.005 | 1.019 | 1.041 | 1.106 | 1.337 |
| | 1/200 | 1.002 | 1.010 | 1.021 | 1.056 | 1.192 |
| $k_{bm}$ | 1/5 | 0.084 | 0.122 | 0.100 | -0.069 | -0.761 |
| | 1/10 | 0.058 | 0.108 | 0.126 | 0.079 | -0.299 |
| | 1/15 | 0.046 | 0.093 | 0.121 | 0.118 | -0.125 |
| | 1/25 | 0.032 | 0.073 | 0.104 | 0.132 | 0.023 |
| | 1/50 | 0.020 | 0.049 | 0.076 | 0.117 | 0.120 |
| | 1/100 | 0.012 | 0.031 | 0.051 | 0.089 | 0.139 |
| | 1/200 | 0.007 | 0.019 | 0.033 | 0.062 | 0.120 |
| $k_{sm}$ | 1/5 | -0.036 | -0.108 | -0.190 | -0.333 | -0.517 |
| | 1/10 | -0.018 | -0.060 | -0.113 | -0.227 | -0.424 |
| | 1/15 | -0.012 | -0.041 | -0.081 | -0.173 | -0.365 |
| | 1/25 | -0.007 | -0.025 | -0.052 | -0.119 | -0.289 |
| | 1/50 | -0.004 | -0.013 | -0.028 | -0.068 | -0.192 |
| | 1/100 | -0.002 | -0.007 | -0.014 | -0.037 | -0.117 |
| | 1/200 | -0.001 | -0.003 | -0.007 | -0.019 | -0.067 |
| $k_{tm}$ | 1/5 | 0.012 | -0.029 | -0.119 | -0.432 | -4.232 |
| | 1/10 | 0.010 | -0.008 | -0.053 | -0.219 | -1.853 |
| | 1/15 | 0.008 | -0.002 | -0.031 | -0.144 | -1.244 |
| | 1/25 | 0.006 | 0.002 | -0.015 | -0.082 | -0.757 |
| | 1/50 | 0.004 | 0.003 | -0.004 | -0.036 | -0.379 |
| | 1/100 | 0.003 | 0.003 | 0.000 | -0.015 | -0.186 |
| | 1/200 | 0.002 | 0.002 | 0.001 | -0.006 | -0.090 |
| $k_{vm}$ | 1/5 | -0.051 | -0.139 | -0.261 | -0.609 | -2.630 |
| | 1/10 | -0.026 | -0.070 | -0.131 | -0.302 | -1.117 |
| | 1/15 | -0.018 | -0.047 | -0.088 | -0.201 | -0.736 |
| | 1/25 | -0.011 | -0.029 | -0.053 | -0.121 | -0.441 |
| | 1/50 | -0.005 | -0.015 | -0.028 | -0.062 | -0.221 |
| | 1/100 | -0.003 | -0.008 | -0.014 | -0.032 | -0.111 |
| | 1/200 | -0.001 | -0.004 | -0.008 | -0.017 | -0.056 |

Table 2. The effect of Poisson's ratio on the stress intensity factor ratios in an isotropic cylindrical shell containing an inclined crack; $\beta = 45°$, $a/h = 3$, $h/R = 1/10$

| $k$ \ $\nu$ | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---|---|---|---|---|---|---|
| $k_{mm}$ | 1.166 | 1.167 | 1.167 | 1.167 | 1.166 | 1.164 |
| $k_{bm}$ | 0.063 | 0.077 | 0.092 | 0.108 | 0.124 | 0.140 |
| $k_{sm}$ | -0.058 | -0.059 | -0.059 | -0.060 | -0.060 | -0.059 |
| $k_{tm}$ | -0.009 | -0.008 | -0.008 | -0.008 | -0.008 | -0.008 |
| $k_{vm}$ | -0.075 | -0.073 | -0.072 | -0.070 | -0.069 | -0.067 |
| $k_{mb}$ | 0.018 | 0.023 | 0.028 | 0.034 | 0.039 | 0.045 |
| $k_{bb}$ | 0.605 | 0.617 | 0.626 | 0.632 | 0.634 | 0.631 |
| $k_{sb}$ | -0.016 | -0.018 | -0.019 | -0.021 | -0.024 | -0.026 |
| $k_{tb}$ | -0.005 | -0.005 | -0.006 | -0.006 | -0.006 | -0.006 |
| $k_{vb}$ | 0.004 | 0.003 | 0.003 | 0.003 | 0.003 | 0.003 |
| $k_{ms}$ | -0.058 | -0.059 | -0.059 | -0.060 | -0.060 | -0.060 |
| $k_{bs}$ | -0.054 | -0.059 | -0.064 | -0.069 | -0.074 | -0.080 |
| $k_{ss}$ | 1.059 | 1.059 | 1.059 | 1.059 | 1.058 | 1.057 |
| $k_{ts}$ | 0.007 | 0.007 | 0.008 | 0.008 | 0.009 | 0.010 |
| $k_{vs}$ | 0.133 | 0.131 | 0.129 | 0.128 | 0.126 | 0.124 |
| $k_{mt}$ | 0.005 | 0.005 | 0.005 | 0.004 | 0.004 | 0.003 |
| $k_{bt}$ | -0.005 | -0.005 | -0.006 | -0.006 | -0.006 | -0.006 |
| $k_{st}$ | -0.007 | -0.007 | -0.007 | -0.006 | -0.006 | -0.005 |
| $k_{tt}$ | 0.309 | 0.325 | 0.339 | 0.353 | 0.366 | 0.379 |
| $k_{vt}$ | -0.095 | -0.094 | -0.093 | -0.091 | -0.090 | -0.088 |
| $k_{mv}$ | -0.223 | -0.244 | -0.266 | -0.287 | -0.308 | -0.330 |
| $k_{bv}$ | -0.004 | 0.001 | 0.007 | 0.014 | 0.022 | 0.032 |
| $k_{sv}$ | -0.174 | -0.187 | -0.200 | -0.213 | -0.226 | -0.238 |
| $k_{tv}$ | 1.138 | 1.166 | 1.191 | 1.213 | 1.233 | 1.250 |
| $k_{vv}$ | 2.304 | 2.287 | 2.272 | 2.258 | 2.244 | 2.231 |

Table 3. The effect of material orthotropy on the stress intensity factor ratios in a cylindrical shell containing an inclined crack; $\beta = 45°$, $a/h = 3$, $h/R = 1/10$

| $k$ \ $E_1/E_2$ | 0.037 | 1.000 | 26.667 |
|---|---|---|---|
| $k_{mm}$ | 1.127 | 1.321 | 1.984 |
| $k_{bm}$ | 0.078 | 0.126 | 0.125 |
| $k_{sm}$ | -0.056 | -0.113 | -0.181 |
| $k_{tm}$ | -0.010 | -0.053 | -0.179 |
| $k_{vm}$ | -0.074 | -0.131 | -0.263 |
| $k_{mb}$ | 0.024 | 0.044 | 0.050 |
| $k_{bb}$ | 0.569 | 0.567 | 0.534 |
| $k_{sb}$ | -0.011 | -0.026 | -0.028 |
| $k_{tb}$ | -0.004 | -0.005 | -0.002 |
| $k_{vb}$ | 0.004 | 0.007 | 0.012 |
| $k_{ms}$ | -0.057 | -0.115 | -0.189 |
| $k_{bs}$ | -0.031 | -0.073 | -0.079 |
| $k_{ss}$ | 1.082 | 1.111 | 1.205 |
| $k_{ts}$ | 0.019 | 0.068 | 0.179 |
| $k_{vs}$ | 0.238 | 0.228 | 0.331 |
| $k_{mt}$ | 0.005 | 0.004 | 0.003 |
| $k_{bt}$ | -0.004 | -0.006 | -0.005 |
| $k_{st}$ | -0.010 | -0.006 | -0.005 |
| $k_{tt}$ | 0.314 | 0.273 | 0.189 |
| $k_{vt}$ | -0.095 | -0.093 | -0.087 |
| $k_{mv}$ | -0.166 | -0.577 | -1.100 |
| $k_{bv}$ | -0.009 | 0.030 | 0.090 |
| $k_{sv}$ | -0.277 | -0.491 | -0.872 |
| $k_{tv}$ | 1.033 | 1.888 | 2.724 |
| $k_{vv}$ | 2.089 | 2.671 | 3.573 |

$dV$ may be expressed as

$$dV = -\int_{-h/2}^{h/2}\int_0^{da}\frac{1}{2}\sum_{j=1}^{3}\sigma_{1j}(0, x_2, x_3)[u_j(+0, x_2-da, x_3) - u_j(-0, x_2-da, x_3)]\,dx_2\,dx_3 \tag{84}$$

where the minus sign is due to the fact that during the "release" of the crack surfaces in $a < x_2 < a + da$, $-h/2 < x_3 < h/2$ the tractions and displacements are in opposite directions (consequently, the total strain energy of the shell decrease). For small values of $da$ we now observe that

$$\sigma_{1j}(0, x_2, x_3) = \frac{k_j(x_3)}{\sqrt{(2(x_2-a))}}, \; (j = 1, 2, 3) \tag{85}$$

$$u_j(+0, x_2-da, x_3) - u_j(-0, x_2-da, x_3) = \frac{4k_j(x_3)}{E}\sqrt{(2(a+da-x_2))}, \; (j = 1,2) \tag{86}$$

$$u_3(+0, x_2-da, x_3) - u_3(-0, x_2-da, x_3) = \frac{k_3(x_3)}{G}\sqrt{(2(a+da-x_2))}, \tag{87}$$

where $k_1$, $k_2$ and $k_3$ are the Modes I, II and III stress intensity factors around the crack border $x_2 = a$.

Referring to the definitions of the stress intensity ratios $k_{ij}$ $(i, j = m, b, s, t, v)$, given by (77)–(81) we can define the "membrane", "bending", "shear", "twisting" and the "transverse shear" components of the stress intensity factors at the crack tip $x_2 = a$ as follows:

$$k_i = \sum_j k_{ij}\sigma_j\sqrt{(a)}, \; (i, j = m, b, s, t, v). \tag{88}$$

From (73)–(75), (77)–(81) and (88) the stress intensity factors may then be expressed as

$$k_1(x_3) = k_m + k_b\left(\frac{x_3}{h/2}\right), \tag{89}$$

$$k_2(x_3) = k_s + k_t\left(\frac{x_3}{h/2}\right), \tag{90}$$

$$k_3(x_3) = k_v\left[1 - \left(\frac{x_3}{h/2}\right)^2\right]. \tag{91}$$

By substituting from (85)–(87) and (89)–(91) into (84) we obtain

$$dV = -\frac{\pi}{E}\left[k_m^2 + \frac{k_b^2}{3} + k_s^2 + \frac{k_t^2}{3} + \frac{4(1+\nu)}{15}k_v^2\right]hda. \tag{92}$$

Observing that $hda = dA$, for the rate of externally added or internally released energy (at one crack tip $x_2 = a$, per unit shell thickness, per unit crack extension in the plane of the original crack) we find

$$\frac{d}{dA}(U - V) = \frac{\pi}{E}\left[k_m^2 + \frac{k_b^2}{3} + k_s^2 + \frac{k_t^2}{3} + \frac{4(1+\nu)}{15}k_v^2\right]. \tag{93}$$

Finally it is again worthwhile to remember that *all* shell theories are, to varying degrees, approximations of the three dimensional elasticity. Therefore, even if the "shallowness" assumption is satisfied, the theory used in this paper and the results given are only approximate. Strictly speaking, the crack problems considered in plates and shells are three dimensional elasticity problems. Such problems in their simplest form do not seem to be as yet analytically tractable. However, from a structural viewpoint, the shell solutions can be useful in the sense that the "plane stress" crack solutions are, that is, the results should be interpreted and used in a certain thickness-average sense. Since the shell theories are quite numerous, there is always the question as to what theory to use in the crack problem. Clearly there is no unique answer for this question. However, one could try to establish some guidelines and set certain minimum requirements. In crack problems the most important information (from an application viewpoint) is embedded in the asymptotic solution of the problem around the crack tips. The first requirement then is that the asymptotic results found from the shell solution must be compatible with that of the in-plane and anti-plane elasticity solutions of the crack problem. This means that the stresses around the crack tips must have the standard square root singularity and their angular distribution must be identical to that given by the related two-dimensional elasticity solutions.

In crack problems since one is interested in the behavior of the solution very near the crack tip, it is natural to assume that all local length parameters would have some influence on the results which are of interest. In a general shallow shell there are five local length parameters, namely three radii of curvature, $R_1$, $R_2$, $R_{12}$, the crack length $2a$, and the thickness $h$. A particular shell theory to be suitable for crack problems should therefore contain four dimensionless (independent) length parameters.

Again, since it is desired that the shell theory give a reasonably accurate solution near the crack tip, it would be necessary that the theory should accommodate all the stress boundary conditions on the crack surfaces separately.

Reissner's transverse shear theory, which has been used in this paper, seems to be the simplest theory which satisfy all these requirements. Aside from a certain degree of confidence one may have in its results, an advantage of such a compatible theory, is that it makes it possible to carry out calculations such as that of energy release rate (see (93)) routinely. This, of course, is primarily due to the fact that the asymptotic results (65)–(67) and (69)–(70) are identical to that of the corresponding elasticity solutions. However, since a higher order shell theory does not necessarily imply higher accuracy in (certain calculated) results, there are still unresolved questions. Are the results of the crack problems obtained from the Reissner's shell tyeory, for example, more reliable than that given by the classical shell theory? For the stress intensity factors we think the answer is yes. The reason for this is largely the fact that the classical theory satisfies none of the requirements listed above. Could one improve the solution further by considering "higher order" theories which may take into account additional features of deformations and stresses (such as, for example, the stretch in thickness direction)? Even if one can solve such problems with the same degree of numerical accuracy as the problems based on simpler shell theories, it would be difficult to know which solution is more reliable. In our view, therefore, it would be very difficult to justify the use of a more complex theory than Reissner's in solving the crack problem in shells unless one has a demonstrable reason for it.

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## APPENDIX A

Dimensionless and normalized quantities used in the analysis.

$$x = x_1/a\sqrt{(c)},\ y = x_2\sqrt{(c)}/a,\ z = x_3/a; \tag{A1}$$

$$u = u_1\sqrt{(c)}/a,\ v = u_2/a\sqrt{(c)},\ w = u_3/a; \tag{A2}$$

$$\beta_x = \beta_1\sqrt{(c)},\ \beta_y = \beta_2/\sqrt{(c)}; \tag{A3}$$

$$\phi(x, y) = F(x_1, x_2)/Eha^2; \tag{A4}$$

$$\sigma_{xx} = \frac{\sigma_{11}}{Ec},\ \sigma_{yy} = \frac{c\sigma_{22}}{E},\ \sigma_{xy} = \frac{\sigma_{12}}{E},\ \sigma_{xz} = \frac{\sigma_{13}}{B\sqrt{(c)}},\ \sigma_{yz} = \frac{\sqrt{(c)}\sigma_{23}}{B}; \tag{A5}$$

$$N_{xx} = N_{11}/Ehc,\ N_{yy} = cN_{22}/Eh,\ N_{xy} = N_{12}/Eh; \tag{A6}$$

$$M_{xx} = M_{11}/Ech^2,\ M_{yy} = cM_{22}/Eh^2,\ M_{xy} = M_{12}/Eh^2; \tag{A7}$$

$$V_x = V_1/Bh\sqrt{(c)},\ V_y = V_2\sqrt{(c)}/Bh; \tag{A8}$$

$$\lambda_1^4 = 12(1-\nu^2)\frac{c^2a^4}{h^2R_1^2},\ \lambda_2^4 = 12(1-\nu^2)\frac{a^4}{c^2h^2R_2^2},$$

$$\lambda_{12}^4 = 12(1-\nu^2)\frac{a^4}{h^2R_{12}^2},\ \lambda^4 = 12(1-\nu^2)\frac{a^2}{h^2},\ \kappa = E/B\lambda^4; \tag{A9}$$

$$E = \sqrt{(E_1/E_2)},\ \nu = \sqrt{(\nu_1\nu_2)},\ (\nu_1/E_1 = \nu_2/E_2),\ c^4 = \frac{E_1}{E_2},\ B = \frac{5E}{12(1+\nu)}. \tag{A10}$$